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3-Sasakian manifolds, 3-cosymplectic manifolds and Darboux theorem

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Dedicated to the memory of Giulio Minervini on the anniversary of his departure.

Abstract

We present a compared analysis of some properties of 3-Sasakian and 3-cosymplectic manifolds. We construct a canonical connection on an almost 3-contact metric manifold which generalises the Tanaka–Webster connection of a contact metric manifold and we use this connection to show that a 3-Sasakian manifold does not admit any Darboux-like coordinate system. Moreover, we prove that any 3-cosymplectic manifold is Ricci-flat and admits a Darboux coordinate system if and only if it is flat.

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1. Introduction

Both 3-Sasakian and 3-cosymplectic manifolds belong to the class of almost contact (metric) 3-structures, introduced by Kuo [13] and, independently, by Udriste [17]. The study of 3-Sasakian manifolds has been conducted by several authors (see for example [5,6] and references therein) due to the increasing awareness of their importance in mathematics and in physics, together with the closely linked hyper-Kählerian and quaternionic Kählerian manifolds. Recently they have made an appearance also in supergravity and M-theory (see [1,2,8]). Less studied, so far, are the 3-cosymplectic manifolds, also called hyper-cosymplectic, but we can list some recent publications [7,12,14,16]. For example, Kashiwada and his collaborators proved in [12] that any b-Kenmotsu (see [4,10]) almost contact 3-structure must be 3-cosymplectic.

In this paper we present a compared analysis of some properties of 3-Sasakian and 3-cosymplectic manifolds. We start with a brief review of some known results on these classes of manifolds, contained in Section 2. In Section 3 we construct a canonical connection on an almost 3-contact metric manifold and we study its curvature and torsion

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analysing also its behaviour in the special cases of 3-Sasakian and 3-cosymplectic manifolds. Our connection can be interpreted as a generalisation of the (generalised) Tanaka–Webster connection of a contact metric manifold, introduced by Tanno in [15]. The section is concluded by a further investigation of the properties of 3-cosymplectic manifolds concerning their projectability which leads us to prove that every 3-cosymplectic manifold is Ricci-flat. In the final section we analyse the possibility of finding a Darboux-like coordinate system on 3-Sasakian and 3-cosymplectic manifolds. Firstly we establish a relation which holds in any almost 3-contact metric manifold linking the horizontal part of the metric with the three fundamental forms Φ_α. This relation is responsible for a kind of rigidity of this class of manifolds which links the existence of Darboux coordinates to the flatness of the manifold and does not hold in the case of a single Sasakian or cosymplectic structure. In particular, on the one hand, using our canonical connection and the (local) projection of a 3-Sasakian manifold over a quaternionic Kählerian manifold (see [5,9]), we show that 3-Sasakian manifolds, unlike the Sasakian ones, do not admit any Darboux-like coordinate system. This result is related to the fact that 3-Sasakian manifolds are not (horizontally) flat. On the other hand, we show that a 3-cosymplectic manifold admits a Darboux coordinate system in the neighbourhood of each point if and only if its metric is flat.

2. Preliminaries

An *almost contact manifold* is an odd-dimensional manifold M which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, where $I: TM \to TM$ is the identity mapping. From the definition it follows also that $\phi\xi = 0$, $\eta \circ \phi = 0$ and that the $(1, 1)$-tensor field ϕ has constant rank $2n$ (cf. [4]). An almost contact manifold (M, ϕ, ξ, η) is said to be *normal* when the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically, $[\phi, \phi]$ denoting the Nijenhuis tensor of ϕ. It is known that any almost contact manifold (M, ϕ, ξ, η) admits a Riemannian metric g such that

$$g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F) \tag{1}$$

holds for all $E, F \in \Gamma(TM)$. This metric g is called a *compatible metric* and the manifold M together with the structure (ϕ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (1), one has $\eta = g(\cdot, \xi)$. The 2-form Φ on M defined by $\Phi(E, F) = g(E, \phi F)$ is called the *fundamental* 2-*form* of the almost contact metric manifold M. Almost contact metric manifolds such that both η and Φ are closed are called *almost cosymplectic manifolds* and almost contact metric manifolds such that $d\eta = \Phi$ are called *contact metric manifolds*. Finally, a normal almost cosymplectic manifold is called a *cosymplectic manifold* and a normal contact metric manifold is said to be a *Sasakian manifold*. In terms of the covariant derivative of ϕ the cosymplectic and the Sasakian conditions can be expressed respectively by

$$\nabla\phi = 0$$

and

$$(\nabla_E \phi)F = g(E, F)\xi - \eta(F)E$$

for all $E, F \in \Gamma(TM)$. It should be noted that both in Sasakian and in cosymplectic manifolds ξ is a Killing vector field.

An *almost* 3-*contact manifold* is a $(4n + 3)$-dimensional smooth manifold M endowed with three almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for every $\alpha, \beta \in \{1, 2, 3\}$,

$$\phi_\alpha\phi_\beta - \eta_\beta \otimes \xi_\alpha = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\phi_\gamma - \delta_{\alpha\beta} I, \qquad \phi_\alpha\xi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\xi_\gamma, \qquad \eta_\alpha \circ \phi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\eta_\gamma, \tag{2}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric symbol. This notion was introduced by Kuo [13] and, independently, by Udriste [17]. In [13] Kuo proved that given an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, there exists a Riemannian metric g compatible with each of them and hence we can speak of *almost contact metric 3-structures*. It is well-known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g and that the structural group of the tangent bundle is reducible to $Sp(n) \times I_3$. Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ one obtains a $4n$-dimensional distribution on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H} \oplus \langle \xi_1, \xi_2, \xi_3 \rangle$. For a reason which will be clearer later we call any vector belonging to the distribution $\mathcal{H}$ "horizontal" and any vector belonging to the distribution $\langle \xi_1, \xi_2, \xi_3 \rangle$ "vertical". An almost 3-contact manifold M is said to be *hyper-normal* if each almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is normal.

When the three structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ are contact metric structures, we say that M is a *3-contact metric manifold* and when they are Sasakian, that is when each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is also normal, we call M a *3-Sasakian manifold*. However these two notions coincide. Indeed as it has been proved in 2001 by Kashiwada [11], every contact metric 3-structure is 3-Sasakian. In any 3-Sasakian manifold we have that, for each $\alpha \in \{1, 2, 3\}$,

$$\phi_\alpha = -\nabla \xi_\alpha. \tag{3}$$

Using this, one obtains that $[\xi_1, \xi_2] = 2\xi_3$, $[\xi_2, \xi_3] = 2\xi_1$, $[\xi_3, \xi_1] = 2\xi_2$. In particular, the vertical distribution $\langle \xi_1, \xi_2, \xi_3 \rangle$ is integrable and defines a 3-dimensional foliation of M denoted by $\mathcal{F}_3$. Since ξ_1, ξ_2, ξ_3 are Killing vector fields, $\mathcal{F}_3$ is a Riemannian foliation. Moreover it has totally geodesic leaves of constant curvature 1. On the contrary, in a 3-Sasakian manifold the horizontal distribution $\mathcal{H}$ is never integrable. About the foliation $\mathcal{F}_3$, Ishihara [9] has shown that if $\mathcal{F}_3$ is regular then the space of leaves is a quaternionic Kählerian manifold. Boyer, Galicki and Mann have proved the following more general result.

Theorem 2.1 (*[5]*). *Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-Sasakian manifold such that the Killing vector fields ξ_1, ξ_2, ξ_3 are complete. Then*

(i) *M^{4n+3} is an Einstein manifold of positive scalar curvature equal to $2(2n + 1)(4n + 3)$.*
(ii) *Each leaf $\mathcal{L}$ of the foliation $\mathcal{F}_3$ is a 3-dimensional homogeneous spherical space form.*
(iii) *The space of leaves $M^{4n+3}/\mathcal{F}$ is a quaternionic Kählerian orbifold of dimension $4n$ with positive scalar curvature equal to $16n(n + 2)$.*

By an *almost* 3-*cosymplectic manifold* we mean an almost 3-contact metric manifold M such that each almost contact metric structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is almost cosymplectic. The almost 3-cosymplectic structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is called 3-*cosymplectic* if it is hyper-normal. In this case M is said to be a 3-*cosymplectic manifold*. However it has been proved recently that these two notions are the same:

Theorem 2.2 (*[7, Theorem 4.13]*). *Any almost 3-cosymplectic manifold is 3-cosymplectic.*

In any 3-cosymplectic manifold we have that ξ_α, η_α, ϕ_α and Φ_α are ∇-parallel. In particular

$$[\xi_\alpha, \xi_\beta] = \nabla_{\xi_\alpha} \xi_\beta - \nabla_{\xi_\beta} \xi_\alpha = 0 \tag{4}$$

for all $\alpha, \beta \in \{1, 2, 3\}$, so that, as in any 3-Sasakian manifold, $\langle \xi_1, \xi_2, \xi_3 \rangle$ defines a 3-dimensional foliation $\mathcal{F}_3$ of M^{4n+3}. However, unlike the case of 3-Sasakian geometry, the horizontal subbundle $\mathcal{H}$ of a 3-cosymplectic manifold is integrable because, for all $X, Y \in \Gamma(\mathcal{H})$, $\eta_\alpha([X, Y]) = -2d\eta_\alpha(X, Y) = 0$ since $d\eta_\alpha = 0$.

3. Further properties of 3-Sasakian and 3-cosymplectic manifolds

In this section we investigate on further properties of 3-Sasakian and 3-cosymplectic manifolds. We start with the following preliminary result.

Lemma 3.1. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost 3-contact metric manifold. Then if M is 3-Sasakian we have, for each $\alpha, \beta \in \{1, 2, 3\}$,*

$$\mathcal{L}_{\xi_\alpha} \phi_\beta = 2 \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \phi_\gamma, \tag{5}$$

and if M is 3-cosymplectic,

$$\mathcal{L}_{\xi_\alpha} \phi_\beta = 0. \tag{6}$$

Proof. For any $X \in \Gamma(\mathcal{H})$ we have, using (3),

$$\begin{aligned}(\mathcal{L}_{\xi_2}\phi_1)X &= \nabla_{\xi_2}(\phi_1 X) - \nabla_{\phi_1 X}\xi_2 - \phi_1\nabla_{\xi_2}X + \phi_1\nabla_X\xi_2 \\ &= (\nabla_{\xi_2}\phi_1)X + \phi_2\phi_1 X - \phi_1\phi_2 X = -2\phi_3 X.\end{aligned}$$

Moreover, we have

$$\begin{aligned}(\mathcal{L}_{\xi_2}\phi_1)\xi_1 &= [\xi_2, \phi_1\xi_1] - \phi_1[\xi_2, \xi_1] = 2\phi_1\xi_3 = -2\xi_3 = -2\phi_3\xi_1, \\ (\mathcal{L}_{\xi_2}\phi_1)\xi_2 &= [\xi_2, \phi_1\xi_2] - \phi_1[\xi_2, \xi_2] = [\xi_2, \xi_3] = 2\xi_1 = -2\phi_3\xi_2, \\ (\mathcal{L}_{\xi_2}\phi_1)\xi_3 &= [\xi_2, \phi_1\xi_3] - \phi_1[\xi_2, \xi_3] = -[\xi_2, \xi_2] - 2\phi_1\xi_1 = 0 = -2\phi_3\xi_3,\end{aligned}$$

from which we conclude that $\mathcal{L}_{\xi_2}\phi_1 = -2\phi_3$. Similarly one can prove $\mathcal{L}_{\xi_3}\phi_1 = 2\phi_2$. Finally, $\mathcal{L}_{\xi_1}\phi_1 = 0$ holds because $(\phi_1, \xi_1, \eta_1, g)$ is a Sasakian structure. The other equalities in (5) can be proved in an analogous way. We now prove (6). For any horizontal vector field X we have

$$(\mathcal{L}_{\xi_\alpha}\phi_\beta)X = \nabla_{\xi_\alpha}(\phi_\beta X) - \nabla_{\phi_\beta X}\xi_\alpha - \phi_\beta(\nabla_{\xi_\alpha}X - \nabla_X\xi_\alpha) = (\nabla_{\xi_\alpha}\phi_\beta)X = 0$$

and, by using (2) and (4), $(\mathcal{L}_{\xi_\alpha}\phi_\beta)\xi_\gamma = [\xi_\alpha, \phi_\beta\xi_\gamma] - \phi_\beta[\xi_\alpha, \xi_\gamma] = 0$. ■

A common property of 3-Sasakian and 3-cosymplectic manifolds is stated in the following lemma.

Lemma 3.2. *Let M be a 3-Sasakian or 3-cosymplectic manifold. Then, for any horizontal vector field X, $[X, \xi_\alpha]$ is still horizontal.*

Proof. $\eta_\beta([X, \xi_\alpha]) = +X(\eta_\beta(\xi_\alpha)) - \xi_\alpha(\eta_\beta(X)) - 2d\eta_\beta(X, \xi_\alpha) = -2d\eta_\beta(X, \xi_\alpha)$, for any $\beta \in \{1, 2, 3\}$. Now, if the structure is 3-cosymplectic $d\eta_\beta = 0$ and if it is 3-Sasakian $d\eta_\beta(X, \xi_\alpha) = g(X, \phi_\beta\xi_\alpha) = \sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\eta_\gamma(X) = 0$ since X is horizontal. ■

Now we attach a canonical connection to any manifold M^{4n+3} with an almost contact metric 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ in the following way. We set

$$\tilde{\nabla}_X Y = (\nabla_X Y)^h, \qquad \tilde{\nabla}_{\xi_\alpha} Y = [\xi_\alpha, Y], \qquad \tilde{\nabla}\xi_\alpha = 0, \tag{7}$$

for all $X, Y \in \Gamma(\mathcal{H})$, where $(\nabla_X Y)^h$ denotes the horizontal component of the Levi-Civita connection. In the following proposition we start the study of the properties of this connection.

Proposition 3.3. *Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost 3-contact metric manifold. Then the 1-forms η_1, η_2, η_3 are $\tilde{\nabla}$-parallel if and only if $d\eta_\alpha(X, \xi_\beta) = 0$ for any $X \in \Gamma(\mathcal{H})$ and any $\alpha, \beta \in \{1, 2, 3\}$. Furthermore $\tilde{\nabla}$ is a metric connection with respect to g if and only if each ξ_α is Killing.*

Proof. Since $\tilde{\nabla}_X Y \in \Gamma(\mathcal{H})$ for any $X, Y \in \Gamma(\mathcal{H})$, we have $(\tilde{\nabla}_X\eta_\alpha)Y = 0$ for all $X, Y \in \Gamma(\mathcal{H})$; moreover, from $\tilde{\nabla}\xi_\beta = 0$ and $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$ it follows also that $(\tilde{\nabla}_E\eta_\alpha)\xi_\beta = 0$, for all $E \in \Gamma(TM)$ and $\alpha, \beta \in \{1, 2, 3\}$. So η_α is $\tilde{\nabla}$-parallel if and only if $(\tilde{\nabla}_{\xi_\beta}\eta_\alpha)X = 0$ for all $\beta \in \{1, 2, 3\}$, i.e. if and only if $\eta_\alpha([\xi_\beta, X]) = 0$ and this is equivalent to requiring that $d\eta_\alpha(X, \xi_\beta) = 0$. Now we prove the second part of the proposition. Firstly, we note that $(\mathcal{L}_{\xi_\alpha}g)(\xi_\beta, \xi_\gamma) = -g([\xi_\alpha, \xi_\beta], \xi_\gamma) - g(\xi_\beta, [\xi_\alpha, \xi_\gamma]) = -2\sum_{\delta=1}^{3}(\epsilon_{\alpha\beta\delta}g(\xi_\delta, \xi_\gamma) + \epsilon_{\alpha\gamma\delta}g(\xi_\beta, \xi_\delta)) = -2(\epsilon_{\alpha\beta\gamma} + \epsilon_{\alpha\gamma\beta}) = 0$, and, by Lemma 3.2, $(\mathcal{L}_{\xi_\alpha}g)(X, \xi_\beta) = \xi_\alpha(g(X, \xi_\beta)) - g([\xi_\alpha, X], \xi_\beta) - g(X, 2\epsilon_{\alpha\beta\gamma}\xi_\gamma) = 0$ for $X \in \Gamma(\mathcal{H})$. Next, we observe that for all horizontal vector fields X, Y, Z, we have

$$\begin{aligned}(\tilde{\nabla}_Z g)(X, Y) &= Z(g(X, Y)) - g((\nabla_Z X)^h, Y) - g(X, (\nabla_Z Y)^h) \\ &= Z(g(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = 0.\end{aligned}$$

Moreover, clearly, $(\tilde{\nabla}_Z g)(X, \xi_\alpha) = 0$. Finally, $(\tilde{\nabla}_E g)(\xi_\alpha, \xi_\beta) = 0$ for any $E \in \Gamma(TM)$ and any $\alpha, \beta \in \{1, 2, 3\}$. So g is $\tilde{\nabla}$-parallel if and only if $(\tilde{\nabla}_{\xi_\alpha}g)(X, Y) = 0$ for any $X, Y \in \Gamma(\mathcal{H})$ and for all $\alpha \in \{1, 2, 3\}$. But, as $\tilde{\nabla}\xi_\alpha = 0$, we have the equality

$$(\tilde{\nabla}_{\xi_\alpha}g)(X, Y) = \xi_\alpha(g(X, Y)) - g([\xi_\alpha, X], Y) - g(X, [\xi_\alpha, Y]) = (\mathcal{L}_{\xi_\alpha}g)(X, Y)$$

from which we get the assertion. ■

In general the canonical connection $\tilde{\nabla}$ is not torsion free. Indeed we have the following result.

Proposition 3.4. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost 3-contact metric manifold. Then the torsion tensor field $\tilde{T}$ of $\tilde{\nabla}$ is given by*

$$\tilde{T}(X,Y) = 2\sum_{\alpha=1}^{3} d\eta_\alpha(X,Y)\xi_\alpha, \qquad \tilde{T}(X,\xi_\alpha) = 0, \qquad \tilde{T}(\xi_\alpha,\xi_\beta) = [\xi_\beta,\xi_\alpha],$$

for all $X, Y \in \Gamma(\mathcal{H})$ and for all $\alpha \in \{1,2,3\}$.

Proof. For any horizontal vector fields X, Y we have

$$\begin{aligned}\tilde{T}(X,Y) &= (\nabla_X Y - \nabla_Y X - [X,Y])^h - [X,Y]^v \\ &= (T(X,Y))^h - \sum_{\alpha=1}^{3} g([X,Y],\xi_\alpha)\xi_\alpha \\ &= 2\sum_{\alpha=1}^{3} d\eta_\alpha(X,Y)\xi_\alpha.\end{aligned}$$

Moreover, it follows from (7) that $\tilde{T}(\xi_\alpha, X) = [\xi_\alpha, X] - [\xi_\alpha, X] = 0$. Finally, for all $\alpha, \beta \in \{1,2,3\}$, we have easily $\tilde{T}(\xi_\alpha,\xi_\beta) = -[\xi_\alpha,\xi_\beta] = [\xi_\beta,\xi_\alpha]$. ■

Corollary 3.5. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost 3-contact metric manifold such that the 1-forms η_1, η_2, η_3 are $\tilde{\nabla}$-parallel. Then, the distribution spanned by ξ_1, ξ_2 and ξ_3 is integrable if and only if $\tilde{T}(E,F) = 2\sum_{\alpha=1}^{3} d\eta_\alpha(E,F)\xi_\alpha$ for all $E, F \in \Gamma(TM)$.*

Proof. From the equality $[\xi_\beta,\xi_\alpha]^v = \sum_{\gamma=1}^{3} \eta_\gamma([\xi_\beta,\xi_\alpha])\xi_\gamma$ it follows that if the distribution spanned by ξ_1, ξ_2 and ξ_3 is integrable, then $\tilde{T}(\xi_\alpha,\xi_\beta) = \sum_{\gamma=1}^{3} \eta_\gamma([\xi_\beta,\xi_\alpha])\xi_\gamma = 2\sum_{\gamma=1}^{3} d\eta_\gamma(\xi_\alpha,\xi_\beta)\xi_\gamma$. The converse is trivial. ■

Actually, the requirement that the Reeb vector fields are parallel, together with Propositions 3.3 and 3.4 uniquely characterise the connection $\tilde{\nabla}$. This is shown in the following theorem.

Theorem 3.6. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost 3-contact metric manifold. Then there exists a unique connection $\tilde{\nabla}$ on M satisfying the following properties:*

(i) $\tilde{\nabla}\xi_1 = \tilde{\nabla}\xi_2 = \tilde{\nabla}\xi_3 = 0$,
(ii) $(\tilde{\nabla}_Z g)(X,Y) = 0$, *for all $X, Y, Z \in \Gamma(\mathcal{H})$,*
(iii) $\tilde{T}(X,Y) = 2\sum_{\alpha=1}^{3} d\eta_\alpha(X,Y)\xi_\alpha$ *and* $\tilde{T}(X,\xi_\alpha) = 0$, *for all $X, Y \in \Gamma(\mathcal{H})$.*

Furthermore, if M is 3-Sasakian, then for all $E, F \in \Gamma(TM)$

$$(\tilde{\nabla}_E\phi_\alpha)F = -\sum_{\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}(\eta_\beta(E)\phi_\gamma F^h - \eta_\gamma(E)\phi_\beta F^h); \tag{8}$$

if M is 3-cosymplectic, then the connection $\tilde{\nabla}$ coincides with the Levi-Civita connection and in particular we have, for each $\alpha \in \{1,2,3\}$, $\tilde{\nabla}\phi_\alpha = 0$.

Proof. The connection defined by (7) satisfies the properties (i)–(iii). Thus we have only to prove the uniqueness of such a connection. Let $\hat{\nabla}$ be any connection on M verifying the properties (i)–(iii). From (i) we get $\hat{\nabla}\xi_\alpha = 0 = \tilde{\nabla}\xi_\alpha$, and, from (iii), $0 = \hat{T}(\xi_\alpha, X) = \hat{\nabla}_{\xi_\alpha}X - \hat{\nabla}_X\xi_\alpha - [\xi_\alpha, X] = \hat{\nabla}_{\xi_\alpha}X - [\xi_\alpha, X]$, which implies that $\hat{\nabla}_{\xi_\alpha}X = [\xi_\alpha, X] = \tilde{\nabla}_{\xi_\alpha}X$ for all $X \in \Gamma(\mathcal{H})$. Thus we have only to verify that $\hat{\nabla}_X Y = \tilde{\nabla}_X Y$ for all $X, Y \in \Gamma(\mathcal{H})$, that is $\hat{\nabla}_X Y = (\nabla_X Y)^h$ for all $X, Y \in \Gamma(\mathcal{H})$. In order to check this equality, we define another connection on M, by setting

$$\bar{\nabla}_E F := \begin{cases} \hat{\nabla}_E F + (\nabla_E F)^v, & \text{for } E, F \in \Gamma(\mathcal{H}); \\ \nabla_E F, & \text{for } E \in \Gamma(\mathcal{H}^\perp) \text{ and } F \in \Gamma(TM); \\ \nabla_E F, & \text{for } E \in \Gamma(TM) \text{ and } F \in \Gamma(\mathcal{H}^\perp), \end{cases}$$

where $(\nabla_E F)^v$ denotes the vertical component of the Levi-Civita covariant derivative. If we prove that $\bar{\nabla}$ coincides with the Levi-Civita connection, then we will conclude that for all $X, Y \in \Gamma(\mathcal{H})$ $\nabla_X Y = \bar{\nabla}_X Y = \hat{\nabla}_X Y + (\nabla_X Y)^v$, from which $\hat{\nabla}_X Y = (\nabla_X Y)^h$. Firstly, note that for all $X, Y \in \Gamma(\mathcal{H})$, using the definition of the Levi-Civita connection ∇ we have

$$\begin{aligned}
\bar{\nabla}_X Y &= \hat{\nabla}_X Y + \sum_{\alpha=1}^{3} g(\nabla_X Y, \xi_\alpha)\xi_\alpha \\
&= \hat{\nabla}_X Y - \frac{1}{2}\sum_{\alpha=1}^{3}(\xi_\alpha(g(X,Y)) - g([\xi_\alpha, X], Y) - g([\xi_\alpha, Y], X) - g([X,Y],\xi_\alpha))\xi_\alpha \\
&= \hat{\nabla}_X Y + \frac{1}{2}\sum_{\alpha=1}^{3}(-(\mathcal{L}_{\xi_\alpha} g)(X,Y) + \eta_\alpha([X,Y]))\xi_\alpha \\
&= \hat{\nabla}_X Y - \sum_{\alpha=1}^{3}\left(\frac{1}{2}(\mathcal{L}_{\xi_\alpha} g)(X,Y) + d\eta_\alpha(X,Y)\right)\xi_\alpha.
\end{aligned}$$

Now we prove that the connection $\bar{\nabla}$ is metric and torsion free. For all $X, Y, Y' \in \Gamma(\mathcal{H})$

$$(\bar{\nabla}_X g)(Y, Y') = X(g(Y,Y')) - g(\hat{\nabla}_X Y, Y') - g(Y, \hat{\nabla}_X Y') = (\hat{\nabla}_X g)(Y, Y') = 0$$

by the preceding equality and the condition (ii). Next, by using (iii), we obtain $\bar{T}(X,Y) = \hat{T}(X,Y) - 2\sum_{\alpha=1}^{3} d\eta_\alpha(X,Y)\xi_\alpha = 0$. Thus $\bar{\nabla}$ coincides with the Levi-Civita connection of M and this implies that $\hat{\nabla} = \tilde{\nabla}$.

Now we prove the second part of the theorem. Assume that M is 3-Sasakian. Then for any $X, Y \in \Gamma(\mathcal{H})$, using (3) and the fact that $\nabla g = 0$ we have

$$\begin{aligned}
(\tilde{\nabla}_X \phi_1)Y &= (\nabla_X \phi_1)Y - \sum_{\alpha=1}^{3} g(\nabla_X(\phi_1 Y), \xi_\alpha)\xi_\alpha + \phi_1\left(\sum_{\alpha=1}^{3} g(\nabla_X Y, \xi_\alpha)\xi_\alpha\right) \\
&= g(X,Y)\xi_1 - \eta_1(Y)X + \sum_{\alpha=1}^{3} g(\phi_1 Y, \nabla_X \xi_\alpha)\xi_\alpha + g(\nabla_X Y, \xi_2)\xi_3 - g(\nabla_X Y, \xi_3)\xi_2 \\
&= -g(\phi_1 Y, \phi_2 X)\xi_2 - g(\phi_1 Y, \phi_3 X)\xi_3 + g(Y, \phi_2 X)\xi_3 - g(Y, \phi_3 X)\xi_2 \\
&= g(Y, \phi_1\phi_2 X)\xi_2 + g(Y, \phi_1\phi_3 X)\xi_3 + g(Y, \phi_2 X)\xi_3 - g(Y, \phi_3 X)\xi_2 = 0.
\end{aligned}$$

Moreover, for any $\alpha, \beta, \gamma \in \{1,2,3\}$, $(\tilde{\nabla}_E \phi_\beta)\xi_\gamma = \tilde{\nabla}_E(\phi_\beta \xi_\gamma) - \phi_\beta \tilde{\nabla}_E \xi_\gamma = \sum_{\alpha=1}^{3} \epsilon_{\alpha\beta\gamma}\tilde{\nabla}_E \xi_\alpha = 0$. Finally, for any $X \in \Gamma(\mathcal{H})$

$$(\tilde{\nabla}_{\xi_1}\phi_1)X = \tilde{\nabla}_{\xi_1}(\phi_1 X) - \phi_1 \tilde{\nabla}_{\xi_1} X = [\xi_1, \phi_1 X] - \phi_1[\xi_1, X] = (\mathcal{L}_{\xi_1}\phi_1)X,$$

so $(\tilde{\nabla}_{\xi_1}\phi_1)X = (\mathcal{L}_{\xi_1}\phi_1)X$. Similarly, one can find $(\tilde{\nabla}_{\xi_2}\phi_1)X = (\mathcal{L}_{\xi_2}\phi_1)X$ and $(\tilde{\nabla}_{\xi_3}\phi_1)X = (\mathcal{L}_{\xi_3}\phi_1)X$. Hence, by applying (5), we have $(\tilde{\nabla}_{\xi_1}\phi_1)X = 0$, $(\tilde{\nabla}_{\xi_2}\phi_1)X = -2\phi_3 X$, $(\tilde{\nabla}_{\xi_3}\phi_1)X = 2\phi_2 X$. Thus, if we decompose any pair of vector fields $E, F \in \Gamma(TM)$ in their horizontal and vertical parts, $E = E^h + \sum_{\alpha=1}^{3}\eta_\alpha(E)\xi_\alpha$ and $F = F^h + \sum_{\alpha=1}^{3}\eta_\alpha(F)\xi_\alpha$, we have

$$\begin{aligned}
(\tilde{\nabla}_E \phi_1)F &= \sum_{\alpha=1}^{3}(\tilde{\nabla}_{\eta_\alpha(E)\xi_\alpha}\phi_1)F^h \\
&= \eta_2(E)(\tilde{\nabla}_{\xi_2}\phi_1)F^h + \eta_3(E)(\tilde{\nabla}_{\xi_3}\phi_1)F^h \\
&= -2\eta_2(E)\phi_3 F^h + 2\eta_3(E)\phi_2 F^h.
\end{aligned}$$

The other equations involving ϕ_2 and ϕ_3 can be proved in a similar way.

Finally, let M be 3-cosymplectic. Then $\nabla_X Y$ is horizontal for every $X, Y \in \Gamma(\mathcal{H})$, since $g(\nabla_X Y, \xi_\alpha) = -g(Y, \nabla_X \xi_\alpha) = 0$ for all $\alpha \in \{1,2,3\}$. Hence, $\nabla_X Y = (\nabla_X Y)^h = \tilde{\nabla}_X Y$. Moreover, $\nabla\xi_\alpha = 0 = \tilde{\nabla}\xi_\alpha$. Finally, $\nabla_{\xi_\alpha} X = \nabla_X \xi_\alpha - [X, \xi_\alpha] = [\xi_\alpha, X] = \tilde{\nabla}_{\xi_\alpha} X$. We conclude that $\nabla = \tilde{\nabla}$. ■

In the next proposition we analyse the curvature of the canonical connection $\tilde{\nabla}$ in a 3-Sasakian manifold.

Proposition 3.7. *Let* $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a* 3*-Sasakian manifold. Then the curvature tensor of* $\tilde{\nabla}$ *verifies* $\tilde{R}_{EF}\xi_\alpha = 0$, $\tilde{R}_{\xi_\alpha\xi_\beta} = 0$ *and* $\tilde{R}_{X\xi_\alpha} = 0$ *for all* $E, F \in \Gamma(TM)$, $X \in \Gamma(\mathcal{H})$ *and* $\alpha, \beta \in \{1, 2, 3\}$. *Moreover, for all* $X, Y, Z \in \Gamma(\mathcal{H})$,

$$\tilde{R}_{XY}Z = (R_{XY}Z)^h + \sum_{\alpha=1}^{3}(d\eta_\alpha(Y, Z)\phi_\alpha X - d\eta_\alpha(X, Z)\phi_\alpha Y). \tag{9}$$

Proof. That $\tilde{R}_{EF}\xi_\alpha = 0$ is obvious since $\tilde{\nabla}\xi_\alpha = 0$. Next, for any $\alpha, \beta \in \{1, 2, 3\}$,

$$\begin{aligned}\tilde{R}_{\xi_\alpha\xi_\beta}E &= \tilde{\nabla}_{\xi_\alpha}[\xi_\beta, E] - \tilde{\nabla}_{\xi_\beta}[\xi_\alpha, E] - \tilde{\nabla}_{2\sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\xi_\gamma}E \\ &= [\xi_\alpha, [\xi_\beta, E]] - [\xi_\beta, [\xi_\alpha, E]] - 2\sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}[\xi_\gamma, E] \\ &= [[\xi_\alpha, E], \xi_\beta] + [[E, \xi_\beta], \xi_\alpha] + [[\xi_\beta, \xi_\alpha], E] = 0\end{aligned}$$

by the Jacobi identity. Moreover, since the distribution $\langle\xi_1, \xi_2, \xi_3\rangle$ is integrable and each ξ_α is Killing, this distribution defines a Riemannian foliation of M^{4n+3}, which can be described, at least locally, by a family of Riemannian submersions. Note that $\tilde{\nabla}$ can be interpreted as the lift of the Levi-Civita connection of the space of leaves. If X, Y are (local) basic vector fields with respect to such a given submersion, then

$$\tilde{R}_{X\xi_\alpha}Y = \tilde{\nabla}_X\tilde{\nabla}_{\xi_\alpha}Y - \tilde{\nabla}_{\xi_\alpha}\tilde{\nabla}_XY - \tilde{\nabla}_{[X,\xi_\alpha]}Y = \tilde{\nabla}_X[\xi_\alpha, Y] - [\xi_\alpha, \tilde{\nabla}_XY] - \tilde{\nabla}_{[X,\xi_\alpha]}Y = 0,$$

since $[\xi_\alpha, Y] = [\xi_\alpha, \tilde{\nabla}_XY] = [X, \xi_\alpha] = 0$ because, as X, Y and $\tilde{\nabla}_XY$ are basic, these brackets are vertical and, by Lemma 3.2, also horizontal, hence they vanish. It remains to prove (9). We have

$$\begin{aligned}\tilde{R}_{XY}Z &= (\nabla_X\tilde{\nabla}_YZ)^h - (\nabla_Y\tilde{\nabla}_XZ)^h - \tilde{\nabla}_{[X,Y]^h}Z - \tilde{\nabla}_{\sum_{\alpha=1}^{3}\eta_\alpha([X,Y])\xi_\alpha}Z \\ &= \left(\nabla_X\left(\nabla_YZ - \sum_{\alpha=1}^{3}\eta_\alpha(\nabla_YZ)\xi_\alpha\right)\right)^h - \left(\nabla_Y\left(\nabla_XZ - \sum_{\alpha=1}^{3}\eta_\alpha(\nabla_XZ)\xi_\alpha\right)\right)^h \\ &\quad - (\nabla_{[X,Y]^h}Z)^h - \sum_{\alpha=1}^{3}\eta_\alpha([X, Y])[\xi_\alpha, Z] \\ &= (R_{XY}Z)^h + \sum_{\alpha=1}^{3}(\eta_\alpha(\nabla_YZ)\phi_\alpha X - \eta_\alpha(\nabla_XZ)\phi_\alpha Y)\end{aligned}$$

from which (9) follows. ■

We will now show that the Ricci curvature of every 3-cosymplectic manifold vanishes. This result is a consequence of the projectability of 3-cosymplectic manifolds onto hyper-Kählerian manifolds which is stated in the following theorem.

Theorem 3.8. *Every regular* 3*-cosymplectic structure projects onto a hyper-Kählerian structure.*

Proof. Since the foliation $\mathcal{F}_3$ is regular, it is defined by a global submersion f from M^{4n+3} to the space of leaves $M'^{4n} = M^{4n+3}/\mathcal{F}_3$. Then the Riemannian metric g projects to a Riemannian metric G on M'^{4n} because each ξ_α is Killing. Moreover, by (6), the tensor fields ϕ_1, ϕ_2, ϕ_3 project to three tensor fields J_1, J_2, J_3 on M'^{4n} and it is easy to check that $J_\alpha J_\beta = \sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}J_\gamma - \delta_{\alpha\beta}I$. In fact (J_α, G) are Hermitian structures which are integrable because $N_\alpha = 0$. ■

Remark 3.9. Without the assumption of the regularity, Theorem 3.8 still holds, but *locally*, in the sense that there exists a family of submersions f_i from open subsets U_i of M^{4n+3} to a $4n$-dimensional manifold M'^{4n}, with $\{U_i\}_{i\in I}$ an open covering of M^{4n+3}, such that the 3-cosymplectic structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ projects under f_i to a hyper-Kählerian structure on M'^{4n}.

Corollary 3.10. *Every* 3*-cosymplectic manifold is Ricci-flat.*

Proof. According to Remark 3.9, let f_i be a local submersion from the 3-cosymplectic manifold M^{4n+3} to the hyper-Kählerian manifold M'^{4n}. Since f_i is a Riemannian submersion, we can apply a well-known formula which relates the Ricci tensors and, M^{4n+3} and M'^{4n} (cf. [7]): for any X, Y basic vector fields

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}'(f_{i*}X, f_{i*}Y) + \frac{1}{2}(g(\nabla_X N, Y) + g(\nabla_Y N, X)) - 2\sum_{i=1}^{n} g(A_X X_i, A_Y X_i) - \sum_{\alpha=1}^{3} g(T_{\xi_\alpha} X, T_{\xi_\beta} Y), \tag{10}$$

where $\{X_1, \ldots, X_{4n}, \xi_1, \xi_2, \xi_3\}$ is a local orthonormal basis with each X_i basic, A and T are the O'Neill tensors associated with f_i, and N is the local vector field on M^{4n+3} given by $N = \sum_{\alpha=1}^{3} T_{\xi_\alpha}\xi_\alpha$. Note that, since the horizontal distribution is integrable, $A \equiv 0$, and by $\nabla\xi_\alpha = 0$ we get $T_{\xi_\alpha}\xi_\alpha = (\nabla_{\xi_\alpha}\xi_\alpha)^h = 0$, $T_{\xi_\alpha}Z = (\nabla_{\xi_\alpha}Z)^v = (\nabla_Z\xi_\alpha + [\xi_\alpha, Z])^v = 0$. Hence the formula (10) reduces to

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}'(f_{i*}X, f_{i*}Y).$$

But $\mathrm{Ric}'(X', Y') = 0$ for all $X', Y' \in \Gamma(TM)$, because M'^{4n} is hyper-Kählerian. Hence $\mathrm{Ric} = 0$ in the horizontal subbundle $\mathcal{H}$. Finally, it is easy to check that $\mathrm{Ric}(\xi_\alpha, \xi_\beta) = 0$ and $\mathrm{Ric}(X, \xi_\beta) = 0$ for any $X \in \Gamma(\mathcal{H})$. ■

4. The Darboux theorem

Let M^{4n+3} be a manifold endowed with an almost contact metric 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. We denote by $\Phi_\alpha^\flat : X \mapsto \Phi_\alpha(X, \cdot)$ the musical isomorphisms induced by the fundamental 2-forms Φ_α between horizontal vector fields and vertical 1-forms. Their inverses will be denoted by $\Phi_\alpha^\sharp$. We also denote by $g_{\mathcal{H}}^\flat$ the musical isomorphism induced by the metric between horizontal vector fields and vertical 1-forms, and by $\phi_\alpha^{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ the isomorphisms induced by the endomorphisms $\phi_\alpha : TM \to TM$.

Lemma 4.1. *In any almost* 3*-contact metric manifold, the following formulas hold, for each* $\alpha \in \{1, 2, 3\}$,

$$g_{\mathcal{H}}^\flat = \Phi_\alpha^\flat \circ \phi_\alpha^{\mathcal{H}}, \qquad \phi_\alpha^{\mathcal{H}} = -\frac{1}{2}\sum_{\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\, \Phi_\beta^\sharp \circ \Phi_\gamma^\flat. \tag{11}$$

Proof. From $\Phi_\alpha(X,Y) = g(X, \phi_\alpha Y)$ we have $-\Phi_\alpha^\flat = g_{\mathcal{H}}^\flat \circ \phi_\alpha^{\mathcal{H}}$. It follows that

$$g_{\mathcal{H}}^\flat = \Phi_\alpha^\flat \circ \phi_\alpha^{\mathcal{H}}, \tag{12}$$

since $\phi_\alpha^2 X = -X + \eta_\alpha(X)\xi_\alpha = -X$ for every $X \in \Gamma(\mathcal{H})$. We now prove the second formula of (11). Since the equation (12) holds for each $\alpha \in \{1, 2, 3\}$, we get

$$\phi_\beta^{\mathcal{H}} \circ \phi_\gamma^{\mathcal{H}} = -\Phi_\beta^\sharp \circ \Phi_\gamma^\flat, \tag{13}$$

for each $\beta, \gamma \in \{1, 2, 3\}$. Moreover, in view of (2), we have $\phi_\beta^{\mathcal{H}} \circ \phi_\gamma^{\mathcal{H}} = \sum_{\alpha=1}^{3} \epsilon_{\alpha\beta\gamma}\phi_\alpha^{\mathcal{H}}$. Thus we obtain $\sum_{\alpha=1}^{3} \epsilon_{\alpha\beta\gamma}\phi_\alpha^{\mathcal{H}} = -\Phi_\beta^\sharp \circ \Phi_\gamma^\flat$, that is $2\phi_\alpha^{\mathcal{H}} = -\sum_{\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\, \Phi_\beta^\sharp \circ \Phi_\gamma^\flat$. ■

Corollary 4.2. *In any almost* 3*-contact metric manifold, the following formula holds in the horizontal subbundle* $\mathcal{H}$,

$$g_{\mathcal{H}}^\flat = -\Phi_1^\flat \circ \Phi_2^\sharp \circ \Phi_3^\flat.$$

Proof. From the two equalities in (11) we obtain

$$g_{\mathcal{H}}^\flat = -\frac{1}{2}\Phi_1^\flat \circ (\Phi_2^\sharp \circ \Phi_3^\flat - \Phi_3^\sharp \circ \Phi_2^\flat).$$

On the other hand, from (13) and (2) we obtain $\Phi_2^\flat \circ \phi_3^{\mathcal{H}} = -\Phi_3^\flat \circ \phi_2^{\mathcal{H}}$. The claim follows. ■

Now we prove that a 3-Sasakian manifold cannot admit any Darboux-like coordinate system. Here for "Darboux-like coordinate system" we mean local coordinates $\{x_1, \ldots, x_{4n}, z_1, z_2, z_3\}$ with respect to which, for each $\alpha \in \{1, 2, 3\}$, the fundamental 2-forms $\Phi_\alpha = d\eta_\alpha$ have constant components and $\xi_\alpha = a_\alpha^1 \frac{\partial}{\partial z_1} + a_\alpha^2 \frac{\partial}{\partial z_2} + a_\alpha^3 \frac{\partial}{\partial z_3}$, a_α^β being functions depending only on the coordinates z_1, z_2, z_3. This is a natural generalisation of the standard Darboux coordinates for contact manifolds.

Theorem 4.3. *Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-Sasakian manifold. Then M^{4n+3} does not admit any Darboux-like coordinate system.*

Proof. Let p be a point of M^{4n+3}. Then in view of Theorem 2.1 there exist an open neighbourhood U of p and a (local) Riemannian submersion f with connected fibres from U onto a quaternionic Kählerian manifold M'^{4n}, such that $\ker(f_*) = \langle \xi_1, \xi_2, \xi_3 \rangle$. Note that the horizontal vectors with respect to f are just the vectors belonging to $\mathcal{H}$, i.e. those orthogonal to ξ_1, ξ_2, ξ_3. Now, suppose by contradiction that about the point p there exists a Darboux coordinate system, that is an open neighbourhood V with local coordinates $\{x_1, \ldots, x_{4n}, z_1, z_2, z_3\}$ as above. We can assume that $U = V$. We decompose each vector field $\frac{\partial}{\partial x_i}$ into its horizontal and vertical components, $\frac{\partial}{\partial x_i} = X_i + \sum_{\alpha=1}^{3} \eta_\alpha(\frac{\partial}{\partial x_i})\xi_\alpha$. Note that

$$\eta_\alpha\left(\frac{\partial}{\partial x_i}\right) = \frac{1}{2}\sum_{\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\, g\left(\frac{\partial}{\partial x_i}, \phi_\beta \xi_\gamma\right) = \frac{1}{2}\sum_{\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\, d\eta_\beta\left(\frac{\partial}{\partial x_i}, \xi_\gamma\right)$$

$$= \frac{1}{2}\sum_{\beta,\gamma,\delta=1}^{3} \epsilon_{\alpha\beta\gamma}\, a_\gamma^\delta\, d\eta_\beta\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial z_\delta}\right), \tag{14}$$

so that $\eta_\alpha(\frac{\partial}{\partial x_i})$ are functions which do not depend on the coordinates x_i. Consequently, the only eventually non-constant components of each horizontal vector field $X_i = \frac{\partial}{\partial x_i} - \sum_{\alpha=1}^{3} \eta_\alpha(\frac{\partial}{\partial x_i})\xi_\alpha$ in the holonomic basis $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{4n}}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3})$ depend at most on the coordinates z_1, z_2, z_3. Actually, for each $i \in \{1, \ldots, 4n\}$, X_i is a basic vector field with respect to the submersion f, thus its components do not depend even on the fibre coordinates z_α, hence they are constant. For proving this it is sufficient to show that, for each $\alpha \in \{1, 2, 3\}$, $[X_i, \xi_\alpha]$ is vertical. Indeed,

$$[X_i, \xi_\alpha] = \sum_{\beta=1}^{3} \frac{\partial a_\alpha^\beta}{\partial x_i}\frac{\partial}{\partial z_\beta} + \sum_{\beta=1}^{3}\left[\eta_\beta\left(\frac{\partial}{\partial x_i}\right)\xi_\beta, \xi_\alpha\right] = \sum_{\beta=1}^{3}\left[\eta_\beta\left(\frac{\partial}{\partial x_i}\right)\xi_\beta, \xi_\alpha\right]$$

because the functions a_α^β do not depend on the coordinates x_i. Then by Corollary 4.2

$$g(X_i, X_j) = -(d\eta_1^\flat \circ d\eta_2^\sharp \circ d\eta_3^\flat)(X_i)(X_j) \tag{15}$$

and so the functions $g(X_i, X_j)$ are constant since each X_i has constant components and the 2-forms $d\eta_\alpha$ are assumed to have constant components, too. The next step is to note that, for all $i, j \in \{1, \ldots, 4n\}$, the brackets $[X_i, X_j]$ are vertical vector fields. We have, by (14),

$$[X_i, X_j] = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] + \sum_{\alpha,\beta=1}^{3}\left[\eta_\alpha\left(\frac{\partial}{\partial x_i}\right)\xi_\alpha, \eta_\beta\left(\frac{\partial}{\partial x_j}\right)\xi_\beta\right]$$

$$- \sum_{\alpha=1}^{3}\left[\eta_\alpha\left(\frac{\partial}{\partial x_i}\right)\xi_\alpha, \frac{\partial}{\partial x_j}\right] - \sum_{\beta=1}^{3}\left[\frac{\partial}{\partial x_i}, \eta_\beta\left(\frac{\partial}{\partial x_j}\right)\xi_\beta\right]$$

$$= 2\sum_{\alpha,\beta,\gamma=1}^{3} \eta_\alpha\left(\frac{\partial}{\partial x_i}\right)\eta_\beta\left(\frac{\partial}{\partial x_j}\right)\epsilon_{\alpha\beta\gamma}\xi_\gamma.$$

Then, for all $i, j, k \in \{1, \ldots, 4n\}$, using (7) and the Koszul formula for the Levi-Civita covariant derivative we obtain

$$2g(\tilde{\nabla}_{X_i} X_j, X_k) = 2g(\nabla_{X_i} X_j, X_k) = X_i(g(X_j, X_k)) + X_j(g(X_k, X_i)) - X_k(g(X_i, X_j))$$
$$- g([X_j, X_k], X_i) + g([X_k, X_i], X_j) + g([X_i, X_j], X_k) = 0,$$

so that $\tilde{\nabla}_{X_i} X_j = 0$. But $\tilde{\nabla}$ projects locally to the Levi-Civita connection ∇' of the quaternionic Kählerian manifold M'^{4n} under the Riemannian submersion f so that in particular we would have that ∇' is flat and this cannot happen because the scalar curvature of M'^{4n}, by Theorem 2.1, must be strictly positive. ■

Now we prove a Darboux theorem for 3-cosymplectic manifolds.

Theorem 4.4. *Around each point of a flat 3-cosymplectic manifold M^{4n+3} there are local coordinates $\{x_1, \ldots, x_n, y_1, \ldots, y_n, u_1, \ldots, u_n, v_1, \ldots, v_n, z_1, z_2, z_3\}$ such that, for each $\alpha \in \{1, 2, 3\}$, $\eta_\alpha = dz_\alpha$, $\xi_\alpha = \frac{\partial}{\partial z_\alpha}$ and, moreover,*

$$\Phi_1 = 2\sum_{i=1}^{n}(dx_i \wedge dy_i + du_i \wedge dv_i) - 2dz_2 \wedge dz_3, \tag{16}$$

$$\Phi_2 = 2\sum_{i=1}^{n}(dx_i \wedge du_i - dy_i \wedge dv_i) + 2dz_1 \wedge dz_3, \tag{17}$$

$$\Phi_3 = 2\sum_{i=1}^{n}(dx_i \wedge dv_i + dy_i \wedge du_i) - 2dz_1 \wedge dz_2, \tag{18}$$

ϕ_1, ϕ_2 and ϕ_3 are represented, respectively, by the $(4n+3) \times (4n+3)$-matrices

$$\phi_1 = \begin{pmatrix} 0 & -I_n & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{19}$$

$$\phi_2 = \begin{pmatrix} 0 & 0 & -I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \tag{20}$$

$$\phi_3 = \begin{pmatrix} 0 & 0 & 0 & -I_n & 0 & 0 & 0 \\ 0 & 0 & -I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

Proof. Let p be a point of M^{4n+3}. Since M^{4n+3} is flat there exists a neighbourhood U of p where the curvature tensor field vanishes identically. Moreover, one can prove by some linear algebra that there exist horizontal vectors $e_1, \ldots, e_n$ such that $\{e_1, \ldots, e_n, \phi_1 e_1, \ldots, \phi_1 e_n, \phi_2 e_1, \ldots, \phi_2 e_n, \phi_3 e_1, \ldots, \phi_3 e_n, \xi_{1_p}, \xi_{2_p}, \xi_{3_p}\}$ is an orthonormal basis of $T_p M$ satisfying the equalities

$$\begin{aligned} &\Phi_1(e_i, \phi_1 e_j) = \delta_{ij}, \quad &&\Phi_1(\phi_2 e_i, \phi_3 e_j) = \delta_{ij}, \quad &&\Phi_1(\xi_{2_p}, \xi_{3_p}) = -1, \\ &\Phi_2(e_i, \phi_2 e_j) = \delta_{ij}, \quad &&\Phi_2(\phi_1 e_i, \phi_3 e_i) = -\delta_{ij}, \quad &&\Phi_2(\xi_{1_p}, \xi_{3_p}) = 1, \\ &\Phi_3(e_i, \phi_3 e_j) = \delta_{ij}, \quad &&\Phi_3(\phi_1 e_i, \phi_2 e_j) = \delta_{ij}, \quad &&\Phi_3(\xi_{1_p}, \xi_{2_p}) = -1, \end{aligned}$$

and such that the values of the 2-forms Φ_α on all the other pairs of basis vectors vanish. Now we define $4n$ vector fields X_i, Y_i, U_i, V_i on U by parallel transport of the vectors e_i, $\phi_1 e_i$, $\phi_2 e_i$, $\phi_3 e_i$, $i \in \{1, \ldots, n\}$. Note that the definition is well-posed because the parallel transport does not depend on the curve. Since the Levi-Civita connection is a metric connection and since $\nabla \xi_\alpha = 0$ we have that $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, U_1, \ldots, U_n, V_1, \ldots, V_n, \xi_1, \xi_2, \xi_3\}$ is an orthonormal frame on U. Moreover by $\nabla \phi_\alpha = 0$ we get that

$$Y_i = \phi_1 X_i, \qquad U_i = \phi_2 X_i, \qquad V_i = \phi_3 X_i, \tag{22}$$

and by $\nabla \Phi_\alpha = 0$ we have

$$\Phi_1(X_i, Y_j) = \delta_{ij}, \qquad \Phi_1(U_i, V_j) = \delta_{ij}, \qquad \Phi_1(\xi_2, \xi_3) = -1, \tag{23}$$

$$\Phi_2(X_i, U_j) = \delta_{ij}, \qquad \Phi_2(Y_i, V_j) = -\delta_{ij}, \qquad \Phi_2(\xi_1, \xi_3) = 1, \tag{24}$$

$$\Phi_3(X_i, V_j) = \delta_{ij}, \qquad \Phi_3(Y_i, U_j) = \delta_{ij}, \qquad \Phi_3(\xi_1, \xi_2) = -1, \tag{25}$$

and the values of the 2-forms Φ_α on all the other pairs of vector fields belonging to the orthonormal frame vanish. Since the vector fields X_i, Y_i, U_i, V_i are, by construction, ∇-parallel we have that the bracket of each pair of these vector fields vanishes identically. This, together with (4) and the vanishing of the brackets $[X_i, \xi_\alpha]$, $[Y_i, \xi_\alpha]$, $[U_i, \xi_\alpha]$ and $[V_i, \xi_\alpha]$ implies the existence of local coordinates $\{x_1, \ldots, x_n, y_1, \ldots, y_n, u_1, \ldots, u_n, v_1, \ldots, v_n, z_1, z_2, z_3\}$ with respect to which

$$X_i = \frac{\partial}{\partial x_i}, \qquad Y_i = \frac{\partial}{\partial y_i}, \qquad U_i = \frac{\partial}{\partial u_i}, \qquad V_i = \frac{\partial}{\partial v_i},$$
$$\xi_1 = \frac{\partial}{\partial z_1}, \qquad \xi_2 = \frac{\partial}{\partial z_2}, \qquad \xi_3 = \frac{\partial}{\partial z_3}.$$

Now, as the 1-forms η_α are closed, they are locally exact, and we have (eventually reducing U) $\eta_\alpha = df_\alpha$ for some functions $f_\alpha \in C^\infty(U)$, and from the relations $\eta_\alpha(X_i) = \eta_\alpha(Y_i) = \eta_\alpha(U_i) = \eta_\alpha(V_i) = 0$, $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$ it follows that $\frac{\partial f_\alpha}{\partial x_i} = \frac{\partial f_\alpha}{\partial y_i} = \frac{\partial f_\alpha}{\partial u_i} = \frac{\partial f_\alpha}{\partial v_i} = 0$, $\frac{\partial f_\alpha}{\partial z_\beta} = \delta_{\alpha\beta}$. Hence, for each $\alpha \in \{1, 2, 3\}$, $\eta_\alpha = dz_\alpha$. Next, by (23)–(25), we get (16)–(18). Finally, by (22) and by $\phi_\alpha \xi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \xi_\gamma$ we deduce that with respect to this coordinate system ϕ_1, ϕ_2 and ϕ_3 are represented by the matrices (19)–(21), respectively. ■

Arguing as in Theorem 4.3 and taking into account that the "vertical" terms $R_{\xi_\alpha \xi_\beta}$ and the "mixed" terms $R_{X \xi_\alpha}$ of the curvature tensor (with $X \in \Gamma(\mathcal{H})$) vanish, one can prove the converse of Theorem 4.4:

Proposition 4.5. *Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-cosymplectic manifold. If each point of M^{4n+3} admits a Darboux coordinate system such that (16)–(18) of Theorem 4.4 hold, then M^{4n+3} is flat.*

Remark 4.6. We conclude noting that in any almost 3-contact metric manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ (and in particular in any hyper-contact manifold (cf. [3])) the metric g is uniquely determined by the three fundamental 2-forms Φ_α and the three Reeb vector fields ξ_α. In particular, in the case of 3-Sasakian manifolds the metric is uniquely determined by the three contact forms η_α. Indeed, on the one hand, it follows from Corollary 4.2 that

$$g(X, Y) = -(d\eta_1^\flat \circ d\eta_2^\sharp \circ d\eta_3^\flat(X))(Y),$$

for any $X, Y \in \Gamma(\mathcal{H})$. On the other hand, we have $g(\xi_\alpha, \xi_\beta) = \delta_{\alpha\beta}$ and $g(X, \xi_\alpha) = \eta_\alpha(X) = 0$. This remark gives an answer to the open problem raised by Banyaga in the Remark 11 of [3].

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